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AN ACYCLICITY THEOREM FOR CELL COMPLEXES IN d DIMENSION

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Let C be a cell complex in d-dimensional Euclidean space whose faces are obtained by orthogonal projection of the faces of a convex polytope in $d+1$ dimensions. For example, the Delaunay triangulation of a finite point set is such a cell complex. This paper shows that the in_front/behind relation defined for the faces of C with respect to any fixed viewpoint x is acyclic. This result has applications to hidden line/surface removal and other problems in computational geometry.

1. Introduction

Suppose you look at two non-intersecting convex bodies in three-dimensional Euclidean space. If one obstructs (part of) the other in your view it cannot also be obstructed. If we extend this observation to a finite collection of mutually disjoint convex objects we get an asymmetric relation which we call the *in_front/behind relation.* Notice that the relation is defined relative to a fixed viewpoint and may change as the viewpoint changes. This relation can be defined for any number of dimensions. For now, we briefly review two algorithmic problems where this relation plays a significant role; it is not surprising that both problems are three-dimensional.

A popular algorithm for *hidden line/surface removal* in computer graphics is the so-called painter's algorithm [7]. It orders the three-dimensional objects from back to front and draws the objects according to this ordering. Hidden parts are thus removed automatically by "overpainting". A cycle in the in_front/behind relation, such as the one shown in Figure 1, contradicts the existence of such an ordering. One way to deal with this difficulty is to cut the objects into smaller pieces so that the relation is acyclic for the pieces [8], but no good bounds on the number of necessary cuts are known.

Another problem where the in_front/behind relation plays a role is the so-called point location problem in three dimension. Given a cell complex with n convex cells, the goal is to store it in some data structure so that, for a later specified query point the cell that contains the point can be determined quickly. For the case where

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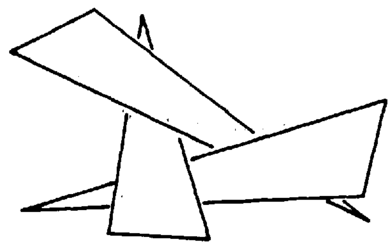

Fig. 1. The three triangles from a cycle in the in_front/behind relation

the above/below relation [1] for the cells is acyclic a data structure that takes $O(m)$ storage and $O(log^2 m)$ query time is given in [2] (here m is the total number of faces of the cell complex). No data structure is known for the general case (for cells with cycles in the relation) that comes even close to this efficient performance.

The main result of this note is that if the objects are the faces of a certain kind of cell complex in E^d, then the in_front/behind relation is acyclic no matter where the viewpoint is chosen. More specifically, the relation is acyclic for all cell complexes in E^d that can be obtained by projecting the boundary complex of a convex polytope in E^{d+1}. For example, the Delaunay triangulation of any finite point set is a cell complex of this kind. A special case of this result (for two-dimensional Delaunay triangulations) was obtained earlier in [6]. Section 2 reviews Delaunay triangulations, related geometric structures, and some of their properties. In Section 3 we prove the acyclicity result for the cells of a d-dimensional Delaunay triangulation. In Section 4 we obtain the general form of the acyclic result. Finally, we offer some remarks in Section 5.

2. Delaunay Triangulations and Related Structures

The most intuitive way to introduce Delaunay triangulations uses so-called Voronoi diagrams. The former are named after Boris Delaunay, also Delone, for his pioneering work in [3] which is dedicated to Georges Voronoi, the namesake for the latter [9].

Let S be a set of n points in E^d. The *Voronoi region* of a point $p \in S$ is the set

$$V(p) = \left\{ x \in E^d \mid \delta(x,p) < \delta(x,q) \text{ for all } q \in S - \{p\} \right\},$$

where δ is the Euclidean distance function. $V(p)$ is the intersection of $n-1$ open half-spaces and thus a convex polyhedron. The *Voronoi diagram* of $S, V(S)$, is the cell complex whose cells are the Voronoi regions of the points is S (see Figure 2). We define the cells and the faces in their boundaries as relatively open sets so that the collection of all faces of $V(S)$, from dimensionality 0 through d, define a partition of

[1] This is the in_front/behind relation for the viewpoint at $(0, 0, \infty)$.

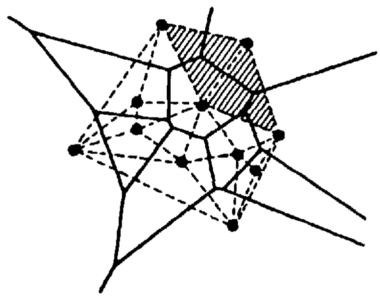

Fig. 2. The Voronoi diagram (solid edges) and the Delaunay triangulation (broken edges) of the same two-dimensional point set are superimposed. Notice that some edges of the Voronoi diagram intersect the corresponding edges of the Delaunay triangulation and some do not

E^d. The *Delaunay triangulation* of $S, D(S)$, is dual to $V(S)$ and contains a $(d-k)$-face for every k-face of $V(S)$ (see Figure 2). More specifically, let F^V be a k-face of $V(S)$ and let $Q \subseteq S$ be maximal so that f^V belongs to the closure of each $V(q)$, $q \in Q$. Then the relative interior of the convex hull for Q, f^D, is a $(d-k)$-face of $D(S)$. If the points of S are in general position (that is, no $d+2$ points lie on a common sphere), then $V(S)$ is a simple cell complex (every k-face is incident to $d-k+1$ $(k+1)$-faces) and $D(S)$ is simplical (every k-face is a k-dimensional simplex). Assuming general position of the points often simplifies things but is not necessary for the result of this paper.

The following properties of Voronoi diagrams and Delaunay triangulations will be important. An *empty sphere* is the boundary of an open ball that is disjoint from S.

Property. Let S be a finite set of points in E^d.

(i) The relative interior of the convex hull of $Q \subseteq S$ is a face of $D(S)$ if and only if there is an empty sphere σ with $Q = S \cap \sigma$.

(ii) Let p^V and q^V be two adjacent vertices of $V(S)$. Then $q^V - p^V$ is normal to the common $d-1$-face of cells p^D and q^D of $D(S)$ (see Figure 2 where a pair p^D, q^D is shaded).

(iii) Let p^D and q^D be two adjacent cells of $D(S)$ and let f^D be the $(d-1)$-face that separates p^D and q^D. Then the points p^V and q^V lie on the line that contains the edge f^V.

Property (iii) sounds like a void statement since, indeed, p^V and q^V are the endpoints of f^V. However, the important point it makes is that the line through f^V is determined solely by the vertices of f^D and thus independent of the other vertices of p^D and q^D. Together with (ii) this will be the key to our main theorem.

Voronoi diagrams and Delaunay triangulations in E^d are related to certain convex polyhedra in E^{d+1} (see [4]). This is best explained using a geometric transform that maps a point $p = (\pi_1, \pi_2, \ldots, \pi_d)$ in E^d to the point

$$p^+ = (\pi_1, \pi_2, \ldots, \pi_d, \pi_1^2 + \ldots + \pi_d^2)$$

in E^{d+1}; we call the transform the *lifting map.* Intuitively, p can be thought of as a point of the hyperplane $x_{d+1} = 0$ in E^{d+1} and p^+ is the vertical projection of p onto the paraboloid $U : x_{d+1} = x_1^2 + x_2^2 + \ldots + x_d^2$. In combination with the lifting map we use a duality transform that maps a point $r = (\rho_1, \rho_2, \ldots, \rho_{d+1})$ in E^{d+1} to the hyperplane

$$r^* : x_{d+1} = 2\rho_1 x_1 + 2\rho_2 x_2 + \ldots + \rho_d x_d - \rho_{d+1},$$

and vice versa, that is, $(r^*)^* = r$. Observe that if r lies on U then r^* is tangent to U and touches U in point r.

Define $P = \{p^+ \mid p \in S\}$ and let $P(S)$ be the convex hull of P. We introduce some notation in order to describe the relation between $P(S)$ and the Delaunay triangulation of S. A point $r = (\rho_1, \rho_2, \ldots, \rho_{d+1})$ is *above* a non-vertical hyperplane $h : x_{d+1} = \eta_1 x_1 + \eta_2 x_2 + \ldots + \eta_d x_d + \eta_{d+1} x_{d+1}$ if

$$\rho_{d+1} > \eta_1 \rho_1 + \eta_2 \rho_2 + \ldots + \eta_d \rho_d + \eta_{d+1};$$

it is said to be *on* the hyperplane if we have equality, and it is *below* the hyperplane if the strict inequality is reversed. A face, f^P, of $P(S)$ is a *lower face* if there is a non-vertical hyperplane that contains the face and no point of $P(S)$ lies below this hyperplane.

Property. (iv) A face f^D belongs to $D(S)$ if and only if there is a lower face f^P of $P(S)$ so that f^D is the vertical projection of f^P onto the hyperplane $x_{d+1} = 0$.

It is rather straightforward to prove property (iv) using property (i) of Delaunay triangulations and the fact that a non-vertical hyperplane intersects U in a (possibly empty) ellipsoid whose vertical projection onto $x_{d+1} = 0$ is a sphere.

Next, set $H = \{(p^+)^* \mid p \in S\}$, a set of non-vertical hyperplanes in E^{d+1}, and define

$$H(S) = \{x \in E^{d+1} \mid x \text{ lies above or on all } h \in H\}.$$

The following relations between the convex polyhedron $H(S)$ and the Voronoi diagram of S can be established (see [4]).

Property. (v) A face f^V belongs to $V(S)$ if and only if there is a face f^H of $H(S)$ so that f^H is the vertical projection of f^H onto $x_{d+1} = 0$.

Of particular importance will be the case of property (v) that relates edges of $V(S)$ with edges of $H(S)$. As in this section, we will use superscripts D, V, P, and H to distinguish faces of $D(S)$, $V(S)$, $P(S)$, and $H(S)$ in the next section. By convention, we let f^D, f^V, f^P, and f^H be faces that related to each other via the transforms explained in this section. For example, if f^D is a vertex of $D(S)$ then f^V is a (d-dimensional) cell of $V(S)$, f^P is also a vertex of $P(S)$, and f^H is a d-dimensional face of $H(S)$.

3. The Acyclicity Theorem for Delaunay Triangulations

We start with a formal definition of the in_front/behind relation. Let x be a point and s and t be two disjoint convex sets in E^d. We write $s \prec_x t$ if there is a half-line ℓ starting at x (but not including x) so that $\ell \cap s \neq \emptyset$, $\ell \cap t \neq \emptyset$, and every point of $\ell \cap s$ lies between x and any point of $\ell \cap t$. If $s \prec_x t$ we say that s is *in front* of t and that t is *behind* s. Notice that it is not necessary that x lie outside s and t. However, if x lies in the interior of s then $s \prec_x u$ for any $u \neq \emptyset$ disjoint from s. The relation is well-defined and asymmetric because s and t are convex.

In order to prove that $\prec_x$ is acyclic for cells of a Delaunay triangulation we introduce a numerical function, Φ_x, so that $\Phi_x(s^D) < \Phi_x(t^D)$ if $s^D \prec_x t^D$. Clearly, if such a function exists then $\prec_x$ is acyclic since it is impossible to have

$$\Phi_x(s_1^D) < \Phi_x(s_2^D) < \ldots < \Phi_x(s_k^D) < \Phi_x(s_1^D).$$

Here is a useful observation that helps establishing the existence of Φ_x with the desired properties. Take two cells s^D and t^D of a Delaunay triangulation so that $s^D \prec_x t^D$. Then there are cells

$$s^D = u_1^D \prec_x u_2^D \prec_x \ldots \prec_x u_j^D = t^D$$

so that u_i^D and u_{i+1}^D are adjacent [2] for $1 \leq i \leq j-1$. For example, the u_i^D can be the cells that intersect the half-line ℓ between s^D and t^D. If ℓ intersects some k-face, for $k \leq d-2$, of the Delaunay triangulation, in which case the u_i^D do not necessarily form a single chain of adjacent cells, we can perturb ℓ ever so slightly (without perturbing its starting point x) so that it still meets s^D and t^D but no face of the triangulation whose dimensionality is less than $d-1$.

Let us now define Φ_x. We perform a translation so that x is the origin of E^d (and E^{d+1}). We define $\Phi_x(s^D)$ as the $d+1^{\text{st}}$ coordinate of s^H — recall that s_H is the vertex of $H(S)$ that corresponds to cell s^D. Algebraically, $\Phi_x(s^D)$ can be defined in terms of the coordinates of $d+1$ points of S: let p_1 through p_{d+1} be $d+1$ affinely independent vertices of s^D and write $p_i = (\pi_{i,1}, \pi_{i,2}, \ldots, \pi_{i,d})$ for $1 \leq i \leq d+1$. Then

$$\Phi_x(s^D) = -\frac{\det\begin{pmatrix} \pi_{1,1} & \cdots & \pi_{1,d} & \pi_{1,1}^2 + \ldots \pi_{1,d}^2 \\ \vdots & \ddots & \vdots & \ddots \\ \pi_{d+1,1} & \cdots & \pi_{d+1,d} & \pi_{d+1,1}^2 + \ldots \pi_{d+1.d}^2 \end{pmatrix}}{\det\begin{pmatrix} \pi_{1,1} & \cdots & \pi_{1,d} & 1 \\ \vdots & \ddots & \vdots & \ddots \\ \pi_{d+1,1} & \cdots & \pi_{d+1,d} & 1 \end{pmatrix}}.$$

Below we show that indeed $\Phi_x(s^D) < \Phi_x(t^D)$ if $s^D \prec t^D$ which implies the acyclicity of $\prec_x$ for the cells of a Delaunay triangulation.

First, we establish the inequality of the following special case. Let s^D and t^D be two adjacent d-simplices of $D(S)$ where p_1 through p_d are the vertices of the

[2] We say that two cells are *adjacent* if they have a $(d-1)$-face in common.

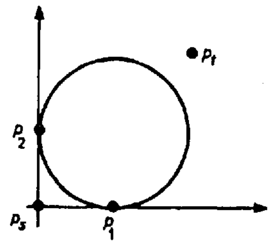

Fig. 3. Points p_s and p_t are chosen so that p_1 and p_2 define an edge of the Delaunay triangulation

common $(d-1)$-face, p_s is the $d+1^{\text{st}}$ vertex of s^D, and p_t is the $d+1^{\text{st}}$ vertex of t^D. We write $p_i = (\pi_{i,1}, \pi_{i,2}, \ldots, \pi_{i,d})$ for $i \in \{1, 2, \ldots, d, s, t\}$ and assume that $\pi_{s,j} = 0$ and $\pi_{t,j} = 2$ for all j, $\pi_{i,i} = 1$ for $1 \leq i \leq d$, and $\pi_{i,j} = 0$ for $1 \leq i \leq d$ and $j \neq i$. Observe that with this choice of points s^D and t^D are indeed cells of $D(S)$, $S = \{p_1, p_2, \ldots, p_d, p_s, p_t\}$, since both p_s and p_t lie outside the sphere with center $(1, 1, \ldots, 1)$ and radius $\sqrt{d-1}$ that goes through p_i for $1 \leq i \leq d$. Furthermore, $s^D \prec_x t^D$ since $x = p_s$ (see Fig. 3). For this special case we have

$$\Phi_x(s^D) = -\frac{\det\begin{pmatrix} 1 & \ldots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 1 \\ 0 & \ldots & 0 & 0 \end{pmatrix}}{\det\begin{pmatrix} 1 & \ldots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 1 \\ 0 & \ldots & 0 & 1 \end{pmatrix}} = -\frac{0}{1} = 0$$

and

$$\Phi_x(t^D) = -\frac{\det\begin{pmatrix} 1 & \ldots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 1 \\ 2 & \ldots & 2 & 4d \end{pmatrix}}{\det\begin{pmatrix} 1 & \ldots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 1 \\ 2 & \ldots & 2 & 1 \end{pmatrix}} = -\frac{4d - 2d}{1 - 2d} = \frac{2d}{2d - 1}.$$

Because $0 < (2d)/(2d-1)$ we thus have established $\Phi_x(s^D) < \Phi_x(t^D)$ for this special case. We next argue that the general case for cells of $D(S)$ can be reduced to the special case by a few simple transformations.

Let s^D and t^D be two arbitrary adjacent cells in $D(S)$ such that $s^D \prec_x t^D$. The first transformation is a rotation about the origin of E^d. Since the paraboloid, U, used to define the lifting map can be obtained by revolution about the x_{d+1}-axis, such a rotation of S and $D(S)$ goes along with a rotation of $P(S)$ and $H(S)$ about the x_{d+1}-axis. We rotate so that the vector $(1, 1, \ldots, 1)$ in E^d is normal to the $(d-1)$-face f^D common to s^D and t^D. Second, we scale the coordinates uniformly so that the hyperplane that contains f^D cuts the positive part of the i^{th} coordinate axis at distance 1 from the origin for $1 \le i \le d$. If we scale the $d+1^{\text{th}}$ coordinate by the same factor this transformation changes only absolute size and retains angels and relative sizes.

Third, we replace f^D by the $(d-1)$-simplex spanned by the intersection points of the hyperplane containing f^D and the d coordinate axes. Although this seems like a drastic change in the geometry of s^D and t^D we will see that it retains all the information that matters. By construction, the vertices of the old f^D lie in the same hyperplane as the vertices of the $(d-1)$-simplex, the new f^D. In terms of the lifting map this means that the lifted points (vertically projected onto U) lie in a common vertical hyperplane of E^{d+1}. If we apply the duality transform to these points we get hyperplanes that have a point at infinity in common (this is the dual point of the vertical hyperplane if we extend the duality transform appropriately). In other words, these hyperplanes are all normal to a common hyperplane. But this implies that the line that is the intersection of the hyperplanes dual to the lifted vertices of the f^D, call it ℓ_{old}, is parallel to ℓ_{new}, the intersection of the hyperplanes dual to the lifted vertices of the new f^D. We orient both lines consistently with $t^H - s^H$ (recall that s^H and t^H are the vertices of $H(S)$ that correspond to s^D and t^D and therefore lie on line ℓ_{old}). Since ℓ_{old} and ℓ_{new} are parallel and consistently oriented the slope of ℓ_{old} is the same as the slope of ℓ_{new}, where "slope" is defined in terms of the increase or decrease in the $d+1^{\text{st}}$ cordinate as we move on the lines.

In the fourth step we replace p_s by the origin and p_t be the point whose coordinates are all equal to 2. This is legitimate because the exact locations of points s^H and t^H on ℓ_{old} are irrelevant as long as s^H lies before t^H (see Property (iii)). In other words, $\Phi_x(s^D) < \Phi_x(t^D)$ if and only if the slope of ℓ_{old} is positive. But the slope of ℓ_{old} is the same as the slope of ℓ_{new} which can be computed by finding the $d+1^{\text{st}}$ coordinate of some two points on ℓ_{new}. Such two points are obtained by using the origin and $(2, 2, \ldots, 2)$ as $d+1^{\text{st}}$ vertices of the transformed s^D and t^D.

This implies a special case of our main theorem (to be formulated in the next section) with which we conclude this section.

Theorem. *The in_front/behind relation defined for the cells of any Delaunay triangulation and for any fixed viewpoint x in E^d is acyclic.*

4. The General Acyclicity Theorem

This section generalizes the above theorem in two directions. First, we show that the in_front/behind relation remains acyclic if we generalize Delaunay triangulations to so-called *regular cell complexes*. These are cell complexes in E^d that can be obtained by orthogonal projection of the boundary complex of a convex polytope in E^{d+1} — of course, only one side of the boundary of the polytope is projected. Second, we show that the in_front/behind relation is acyclic not only for the cells (the d-dimensional faces) of regular cell complexes but also for the lower-dimensional faces.

To show the first generalization we do one more simple transformation to reduce the problem to Delaunay triangulations. Let P be a convex polytope in E^{d+1} and let C be the d-dimensional cell complex obtained by vertically projecting all lower faces of P onto the hyperplane $x_{d+1} = 0$. Since P is a convex polytope, we can define H as in Section 2: if Q is the set of vertices of P then

$$H = \{x \in E^{d+1} \mid x \text{ is above or on hyperplane } p^*, \text{ for every } p \in Q\}.$$

As before, every k-face of P corresponds to a $(d-k)$-face of H and the vertical projection of two corresponding faces are orthogonal to each other. The numerical function we use to prove acyclicity is the same as before: move the viewpoint x to the origin and set $\phi_x(s^C)$ equal to the $d+1^{\text{st}}$ coordinate of vertex s^H of H. Below we argue that we still have $\Phi_x(s^C) < \Phi_x(t^C)$ if $s^C \prec_x t^C$.

Take two adjacent cells $s^C \prec_x t^C$ of C and let f^C be the $(d-1)$-face that is common to s^C and t^C. We assume that s^C and t^C are d-simplices and that f^C is a $(d-1)$-simplex. Otherwise, substitute simplices for s^C, t^C and f^C which are spanned by appropriately chosen vertices of s^C and t^C. Let s^P, t^P and f^P be the corresponding faces of P. We have $\Phi_x(s^C) < \Phi_x(t^C)$ if we can show that the directed line that goes through points s^H and t^H in this order has positive slope; this line, ℓ_{old}, contains the edge f^H. To see that this is true translate the vertices of f^P vertically so that they all lie on the paraboloid $U : x_{d+1} = x_1^2 + x_2^2 + \ldots + x_d^2$. (If necessary, we also translate the other vertices of s^P and t^P so that the dihedral angle at f^P remains convex.) These translations leave s^C, t^C, and f^C unchanged. By definition of the duality transform, the vertical translation of a point p in E^{d+1} corresponds to a vertical translation of the dual hyperplane p^*. This implies that the line ℓ_{old} is parallel to ℓ_{new} that goes through the dual images of the new s^P and t^P, in this order. By construction, ℓ_{new} is the intersection of d hyperplanes all of which are tangent to U. By the argument in Section 3, the slope of ℓ_{new} is therefore positive. Thus, $\Phi_x(s^C) < \Phi_x(t^C)$ since ℓ_{new} and ℓ_{old} are parallel and consistently oriented. This is what we started out to prove.

Second, we argue that the in_front/behind relation of the lower-dimensional faces of a projective cell complex is also acyclic. Let C be such a complex in E^d and assume it is not true. Thus, there is a cycle

$$f_1^C \prec_x f_2^C \prec_x \ldots \prec_x f_k^C \prec_x f_{k+1}^C = f_1^C$$

of faces of C.

The argument is easy if we assume general position of x, that is, x does not lie in the affine hull of any k-face [3] of C for $0 \leq k \leq d-1$. In this case we can replace each face f_i^C by the cell that lies immediately in front of it as viewed from x. The relations are inherited from the f_i^C's which thus gives a cycle of cells and a contradiction to the above result.

[3] The *affine hull* of a k-face is the unique affine k-dimensional subspace that contains it.

What is the difficulty if we remove the general position assumption? We can no longer replace each face that is not a cell by the cell immediately in front, because this cell is not necessarily defined. Here is a way around this difficulty. Let x' be a new viewpoint sufficiently close to x that is in general position as defined above. Every f_i^C which is not a cell can now be replaced by the cell c_i^C that lies immediately in front of f_i^C as viewed from x'. We define $c_i^C = f_i^C$ if the latter is a cell. Thus we get a sequence

$$c_1^C, c_2^C, \ldots, c_k^C, c_{k+1}^C = c_1^C$$

of cells, and to reach a contradiction we just need to show that $c_i^C \prec_{x'} c_{i+1}^C$ for $1 \leq i \leq k$. This is obviously true if $f_i^C \prec_{x'} f_{i+1}^C$. Otherwise, f_i^C, f_{i+1}^C, and x must lie in a common hyperplane that avoids x'. Let ℓ_x be the half-line that starts at x and intersects the relative interiors of f_i^C and f_{i+1}^C, in this order, and let $\ell_{x'}$ be the half-line parallel to ℓ_x that starts at x'. Because x' is sufficiently close to x, $\ell_{x'}$ intersects both c_i^C and c_{i+1}^C, in this order, which implies $c_i^C \prec_{x'} c_{i+1}^C$. Thus, we have a cycle of cells and a contradiction.

We conclude with the main result of this paper which summarizes what has been said in this section.

Main Theorem. *The in_front/behind relation defined for the faces of any regular cell complex and for any fixed viewpoint x in E^d is acyclic.*

The acyclicity of the faces implies that every regular cell complex can be shelled[4] so that at any point in time the union of the faces shelled so far is star-shaped. Furthermore, for any point x there exists such a shelling such that x is always in the kernel of this evolving star-shaped polyhedron.

Remarks

The most important application of the main theorem is probably in three dimensions where it implies that the faces of any regular cell complex can be painted from back to front without creating inconsistencies. Since Delaunay triangulations (both closest-point and furthest-point) are regular cell complexes, this is in particular true for so-called α-shapes of a point set (for a given real number α, the α-shape of a finite point set is a cell complex defined by a subset of all faces of either Delaunay triangulation of the set). In two dimensions, α-shapes are studied in [5] and [4]; extensions to three and higher dimensions are straightforward.

As noticed in [1], regular cell complexes can be viewed as power diagrams defined by weighted points. Using this interpretation it is straightforward to prove a weaker version of our main theorem (one where only one side of a hyperplane through the viewpoint is considered). The idea is to sort the cells according to the x_d-coordinates of the points if the viewpoint is at infinity in the direction of the negative x_d-axis. Otherwise, one can do a projective transformation that maps the problem into this

[4] A *shelling* is a sequence of the faces of the complex so that the union faces of any prefix of the sequence is topologically a ball.

case. However, if all parts of the cell complex are treated at once (which is necessary for a full cyclic view of a scene) then this approach breaks down.

Note added to proof. Raimund Seidel and Bernd Sturmfels observed that the main theorem of this paper can be established using line shellings of the polytope P. We also mention that after completion of this paper Franco Preparata and Roberto Tamassia designed a data structure for three-dimensional point location that is efficient even in the presence of cycles in the above/below relation of the cells.

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